M.Math. IInd year
Mid-semestral exam - IInd semester 2019
Algebraic Number Theory - B.Sury
ANSWER ANY SIX INCLUDING QUESTION 7.

Q 1.
Let K be an algebraic number field.
(a) Prove that every $x \in K$ can be written as a/b with $a \in O_K$ and $b \in \mathbb{Z}$.
(b) Show that each non-zero prime ideal of O_K contains a unique prime number.

OR

Prove that $P \neq P^2$ for any non-zero prime ideal.

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Q 2.
Find all the ideals of $\mathbf{Z}[\sqrt{-6}]$ which contain 6. Which of these are prime ideals?

OR

In the ring $\mathbf{Z}[\sqrt{-5}]$, find with proof a positive integer a such that $(a, 1+2\sqrt{-5})(3, 1+2\sqrt{-5}) = (1+2\sqrt{-5})$.

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Q 3.
(a) For fractional ideals I, J in a Dedekind domain A, prove that $I \supseteq J$ if and only if, $IQ = J$ where Q is an ideal of A.
(b) Define an embedding of O_K as a lattice in $\mathbb{R}^n$ where $[K : \mathbb{Q}] = n$. Compute the covolume of this lattice.

OR

(a) Determine with proof the sign of the discriminant of a number field.
(b) Let K be a cubic extension of $\mathbf{Q}$ such that $-49 \leq \text{disc}\ (K) < 0$. Use the Minkowski bound to deduce that K has class number 1.

Q 4.
Let $K = \mathbb{Q}(\zeta)$ be a cyclotomic field where ζ is a primitive N-th root of unity. If $N = p^k$ for an odd prime p and $k \geq 1$, determine the trace of ζ^{p^m} for any $m \leq k$.

OR

Let $K = \mathbb{Q}(\zeta)$ where ζ is a primitive $2^n + 1$-th root of unity and let $\Phi(X)$ denote the minimal polynomial of ζ over $\mathbb{Q}$. Consider the reduction $\overline{\Phi}(X)$ of $\Phi(X)$ mod 2. Determine the number of irreducible factors of the polynomial $\overline{\Phi}(X)$ and their degrees. You may assume the Kummer-Dedekind criterion.

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Q 5.
Let α be a root of $f(X) = X^5 - X - 1$ and let $K = \mathbf{Q}(\alpha)$. Assume that the polynomial f is irreducible and the discriminant of K is 19×151. Show that the ideal $(19, \alpha + 6)^2$ divides $19O_K$.

OR

For any Galois extension L/K of number fields and prime ideal P of $\mathcal{O}_L$, show that the decomposition group at P surjects onto the Galois group of the residue field extension.

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Q 6.
(a) Determine the class group of $\mathbb{Q}(\sqrt{6})$.
(b) Prove that the class group of $\mathbb{Q}(\sqrt{-23})$ has order 3.

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Q 7.
Describe an embedding of the unit group O_K^* in $\mathbb{R}^{r+s}$ and show that its rank is at the most $r + s - 1$. Further, determine the fundamental unit of the cubic field $\mathbb{Q}(2^{1/3})$.